Saturation of Fidelity in the Atom-Optics Kicked Rotor

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Abstract.
We show that the quantum fidelity is accessible to cold atom experiments for a large class of evolutions in periodical potentials, properly taking into account the experimental initial conditions of the atomic ensemble. We prove analytically that, at the fundamental quantum resonances of the Atom-Optics Kicked Rotor, the fidelity saturates at a constant, time-independent value after a small number of kicks. The latter saturation arises from the bulk of the atomic ensemble, whilst for the resonantly accelerated atoms the fidelity is predicted to decay slowly according to a power law.

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1. Introduction

One of the most intriguing quantum phenomena is the coherent evolution and control of a superposition of state vectors, which is at the heart of quantum parallelism in quantum computation schemes. One way to directly probe coherent quantum evolution is offered by the fidelity, which correlates the time evolution of a reference system with its slightly perturbed dynamics through $F(t) = \left|\langle\psi_0|e^{\mathrm{i}\hat{H}_\Delta t/\hbar}e^{-\mathrm{i}\hat{H}t/\hbar}|\psi_0\rangle\right|^2$. Here, $\hat{H}$ and $\hat{H}_\Delta = \hat{H} + \Delta\hat{V}$ represent the reference Hamiltonian and its perturbed variant, respectively, which generate the evolution, starting from the same initial state $|\psi_0\rangle$. $F(t)$ was introduced as a measure for the stability of quantum motion with respect to changes in some control parameter Δ [1]. The fidelity has recently evoked considerable interest as an alternative route for the study of the effects of perturbations on the coherent evolution of quantum systems, in particular in the context of quantum-classical correspondence [2] and of quantum information processing [3]. As for the former, the finite size of $\hbar$ makes the classical definition of chaos through the exponential divergence of initially nearby trajectories meaningless on the quantum level. However, the sensitivity to changes in Δ prevails both in the classical as well as in the quantum realm, and is dynamically quantified by $F(t)$.

Whilst previous studies focused on the limit of small effective Planck constant [2, 4], we present in this Letter an analytic result which is valid (a) in the *deep quantum*

regime, and (b) for *arbitrary* values of the control parameter Δ. We study the fidelity for cold, periodically kicked atoms, which mimic a paradigmatic model of quantum chaos theory, the δ-Kicked Rotor (KR) [5]. In this context, we show that bona fide fidelity is readily accessible, yet has never been measured in an experiment such as reported in [6]. Specifically, we derive an analytic expression for the fidelity at the fundamental quantum resonances of the KR. At these resonances, a subclass of the experimental atomic ensemble [7, 8, 9] experiences an enhanced acceleration by the kicking field, at precisely defined kicking periods [5]. We show that this resonantly driven subclass induces a slow algebraic decay of fidelity, whilst the bulk of the initial atomic ensemble produces the novel phenomenon of a perfect saturation of fidelity at a finite and experimentally easily detectable level.

Quantum resonances are of purely quantum origin, although it has been recently shown that they can be mapped onto a pseudo-classical problem. In this sense, we study the quantum fidelity in the deep quantum regime, far from the semiclassical limit of large actions, but close to the *pseudo-classical* limit. The latter anchors the near-resonant quantum dynamics to elliptic, stable resonance islands in the phase space of a suitably defined classical map, precisely alike the usual semiclassical limit. However, the pseudoclassical limit is reached at fixed and finite $\hbar$ [10]. Notwithstanding this formal complication, the correspondence between (pseudo)classically stable dynamics and near-resonant quantum evolution immediately suggests the remarkable robustness of the quantum dynamics at resonance, originally believed to be very sensitive to perturbations (see [11] for a further discussion). Hence, quantum resonances provide a novel and *robust* tool of quantum control, and fall within a larger class of particularly robust modes originating from the dynamics of strongly perturbed quantum systems [10, 12]. This robustness manifests indeed in the temporal evolution of fidelity as shown in the sequel.

2. The Atom-Optics Kicked Rotor and Quantum Resonant Motion

The experimental realization [7, 8] of the KR uses cold atoms periodically kicked by an optical lattice. The Hamiltonian that generates the evolution of the atomic wave function in its centre-of-mass degree of freedom is given in the following dimensionless form [13]

$$\hat{H}(t) = \frac{\hat{P}^2}{2} + k\cos(X) \sum_{m=-\infty}^{+\infty} \delta(t - m\tau) \; , \tag{1}$$

where $\hat{P}$ is the momentum operator, and X the atom's position along the pulsed optical lattice. The two parameters that control the dynamics are the kicking period τ, which takes the role of an effective Planck constant in our units [5], and the kick strength k. The Hamiltonian (1) describes the Atom-Optics Kicked Rotor (AOKR) with the atoms moving along a line rather than on a circle (as in the standard KR model). (1) induces the characteristic properties of the quantum KR, which sensitively depend on the parameter τ [5, 10]. If $\tau = 4\pi r/q$, with r, q integers, then the kicking period is

rationally related to the propagation time of the kicked atoms across the lattice constant (which is 2π in our units), and quantum transport is enhanced. Otherwise classically diffusive energy growth is suppressed by dynamical localisation [5]. In the following, we derive analytical results for the fidelity, for the experimentally relevant quantum resonances at $\tau = 2\pi\ell$ ($\ell \in \mathbb{N}$) [8], *i.e.*, for *large* effective Planck constants τ.

An atom subject to a potential periodic in space and time is described by a wave packet composed of 2π-periodic Bloch states, $\psi_\beta(x + 2\pi) = \psi_\beta(x)$, with the quasimomentum β. In our units, the latter is given by the fractional part of the momentum $p = n + \beta, n \in \mathbb{N}$. β is a constant of motion, so the different Bloch states evolve independently of one another, and their momenta only change by integers. At $\tau = 2\pi\ell$, a special behaviour occurs only for a specific, small *subclass* of quasimomenta β. In much the same way as spatial periodicity enforces ballistic motion in physical space, the one-period evolution operator is periodic in momentum space, enforcing ballistic propagation of the corresponding β-states in momentum space. As shown in [10], during one period τ, $|\psi_\beta\rangle$ evolves into $\hat{\mathcal{U}}_\beta\,|\psi_\beta\rangle$, with

$$\hat{\mathcal{U}}_\beta \;=\; e^{-\mathrm{i}\,k\cos(\theta)}\; e^{-\mathrm{i}\,\frac{\tau}{2}(\hat{\mathcal{N}}+\beta)^2}, \tag{2}$$

where $\theta = X\mathrm{mod}(2\pi)$, and $\hat{\mathcal{N}} = -\,\mathrm{i}\;d/d\theta$ is the angular momentum operator. For instance, at $\tau = 4\pi$, and $\beta = 0, 1/2$, or at $\tau = 2\pi$ and $\beta = 1/2$, the second factor of $\hat{\mathcal{U}}_\beta$ equals the identity (apart from a global phase for $\beta \neq 0$), and the energy of the corresponding β-state grows ballistically, *i.e.*, quadratically with the number N of kicks [5]. For general values of β, but $\tau = 2\pi\ell$, ℓ integer, the state after the Nth kick is, in the momentum representation [10]

$$\langle n\,|\hat{\mathcal{U}}_\beta^N\psi_\beta\rangle = e^{\mathrm{i}\,n\arg(W_N)}\int_0^{2\pi}\frac{d\theta}{\sqrt{2\pi}}e^{-\,\mathrm{i}n\theta-\,\mathrm{i}k|W_N|\cos(\theta)}\psi_\beta\left(\theta - N\xi - \arg(W_N)\right)\;, \tag{3}$$

where $W_N = W_N(\xi) := \sum_{s=0}^{N-1}\exp(-\,\mathrm{i}\,s\xi)$. If the initial state of the atom is a plane wave of momentum $p_0 = n_0+\beta$, then ξ takes the constant value $\xi_0 = \pi\ell(2\beta-1)$. If $\xi_0 \neq 0$, then $|W_N| = |\sin(N\xi_0/2)/\sin(\xi_0/2)|$, else if $\xi_0 = 0$ then $|W_N| = N$, and the average kinetic energy increases like $k^2N^2/4$ [5]. The latter occurs at the resonant values $\beta = 1/2 + n/\ell$ mod(1), $n = 0, 1, .., \ell - 1$, which support *ballistic* motion [10]. The remaining Bloch components of the original wave packet, with β not belonging to the resonant subclass, undergo a quasiperiodic energy exchange with the kicking field, leading to a *finite* spread of the associated momentum distribution for all times [10]. How does a perturbation in the kick strength k affect such quantum dynamics at quantum resonance – brought about by the abovementioned complete phase revival in-between two successive kicks? The answer to this question will be given by computing the fidelity of the resonant time evolution, using eq. (3) for two different values of k. Before doing so, we show how to extract the fidelity from observables directly measurable in the experiment [6].

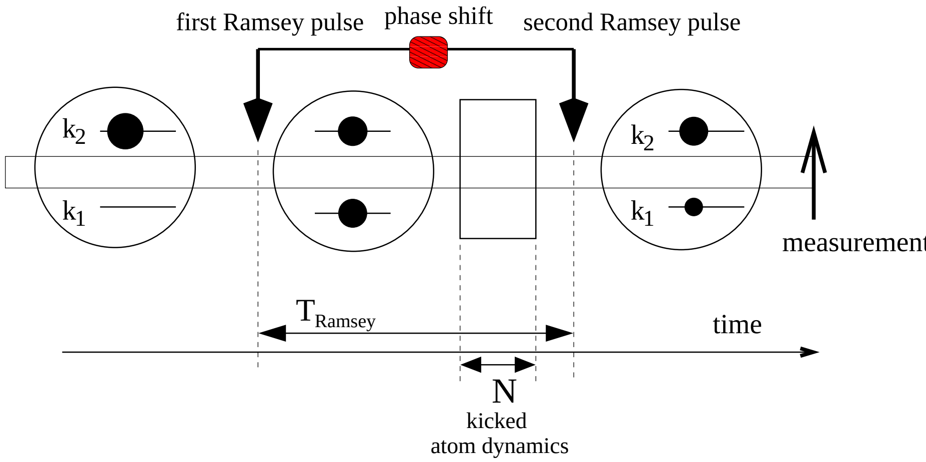

Figure 1. Experimental setup of [6]. Each atom is prepared in the upper state (left), then a $\pi/2$ Ramsey pulse is applied. The dynamics of the coherent superposition of the upper (where $k = k_2$) and lower ($k = k_1$) level lasts for N kicks, before a second $\pi/2$ Ramsey pulse is applied (with time delay $T_{\rm RA}$ with respect to the first one). In principal, any type of dynamics may occur in-between the two Ramsey pulses, but our analytical results restrict here to the AOKR, as an example for a coherent atomic evolution in a periodical optical lattice.

3. Experimental Measurement of Fidelity

The experiment uses two sublevels of the atomic ground state of the electronic degree of freedom, which couple differently to the kick potential. Before the kicks are applied, the sublevels are equally populated, and then evolve coherently. The splitting of the population is performed by a Ramsey-type interferometer technique [6, 14], and the complete time evolution of the atoms is sketched in Fig. 1. The internal states are represented by $(\psi_\beta^{(1)}, \psi_\beta^{(2)})$, where the upper index denotes the corresponding electronic level of the atom. E_1, E_2 are the energies of the two levels. The Ramsey pulses are tuned to the electronic transition between these two levels, *i.e.*, they have a frequency $\omega_{\rm RA} \simeq (E_2 - E_1)/\hbar$. For simplicity, we assume instantaneous Ramsey pulses which occur immediately before the first and after the last kick, respectively. These pulses couple the two states whilst they evolve coherently, but independently, under the kicking potential. Let $\hat{\mathcal{U}}_\beta^{(1)}$ and $\hat{\mathcal{U}}_\beta^{(2)}$ be the propagators (2) in the upper and in the lower atomic level, respectively, and ϕ the relative phase of the two Ramsey pulses. For an initial state where all the population resides in one of the electronic sublevels, *i.e.*, $\psi_\beta^{(1)} = 0$, $\psi_\beta^{(2)} = \psi_\beta$, the states after the final Ramsey pulse are [11]

$$\psi_\beta^{(1)}(N,\phi) = \frac{1}{2} e^{-\mathrm{i} E_1 N} \left(\left(\hat{\mathcal{U}}_\beta^{(1)}\right)^N e^{-\mathrm{i}\phi_{\rm RA}} + \left(\hat{\mathcal{U}}_\beta^{(2)}\right)^N \right) \psi_\beta$$

$$\psi_\beta^{(2)}(N,\phi) = \frac{1}{2} e^{-\mathrm{i} E_2 N} \left(- \left(\hat{\mathcal{U}}_\beta^{(1)}\right)^N e^{\mathrm{i}\phi_{\rm RA}} + \left(\hat{\mathcal{U}}_\beta^{(2)}\right)^N \right) \psi_\beta \ ,$$

with the phase $\phi_{\rm RA} \equiv (E_2 - E_1)N - \phi$. Projection onto a momentum eigenstate with momentum n gives the momentum distribution, and we obtain the total probability in

state $\psi_\beta^{(1)}$ upon summation over all n:

$$P_1(N,\phi_{\rm RA}) = \frac{1}{2}\left[1 + {\rm Re}\left\{e^{-{\rm i}\phi_{\rm RA}}\langle\left(\hat{\mathcal{U}}_\beta^{(2)}\right)^N\psi_\beta|\left(\hat{\mathcal{U}}_\beta^{(1)}\right)^N\psi_\beta\rangle\right\}\right] . \tag{4}$$

The observable given by eq. (4) can be accessed experimentally by measuring the atomic momentum distribution. The measurement, however, provides only an *average over all* β in the initial ensemble [6]. Thus, when the initial pure state ψ_β (of the atomic centre-of-mass motion) is replaced by a mixture which is described by the statistical density operator $\hat{\rho} = \int_0^1 d\beta\, f(\beta)|\psi_\beta\rangle\langle\psi_\beta|$, the final, measurable population in state $\psi_\beta^{(1)}$ is

$$\overline{P}_1(N,\phi_{\rm RA}) = \frac{1}{2}\left\{1 + \sqrt{F(N)}\cos(\phi'')\right\} . \tag{5}$$

We defined [11]

$$\sqrt{F(N)}\cos(\phi'') \equiv \int_0^1 d\beta f(\beta){\rm Re}\left\{e^{-{\rm i}\phi_{\rm RA}}\langle\left(\hat{\mathcal{U}}_\beta^{(2)}\right)^N\psi_\beta|\left(\hat{\mathcal{U}}_\beta^{(1)}\right)^N\psi_\beta\rangle\right\} , \tag{6}$$

where, at fixed time N, ϕ'' differs from $\phi_{\rm RA}$, as well as from ϕ by a constant shift, and

$$F(N) = \left|\int_0^1 d\beta\, f(\beta)\langle\left(\hat{\mathcal{U}}_\beta^{(1)}\right)^N\psi_\beta|\left(\hat{\mathcal{U}}_\beta^{(2)}\right)^N\psi_\beta\rangle\right|^2 \tag{7}$$

corresponds to the fidelity for an incoherent ensemble of β-states. From this analysis, we see that the fidelity is equal to the squared difference between the maximum and the minimum values taken by $\overline{P}_1(N,\phi)$, while the original phase ϕ varies in $[0,2\pi)$ [11, 15].

Hence, the fidelity is accessible as the visibility of the integrated momentum distribution (5) as a function of the Ramsey phase ϕ by the experimental setup of Fig. 1. One must, however, explicitly take into account the scan of the Ramsey phase, in contrast to what has been dubbed "fidelity" in [6]. Moreover, as long as the evolution operators are known and the atoms evolve in a spatially periodical potential, eqs. (4-7) remain valid and are not restricted to the here analysed AOKR.

4. Analytical Results and Discussions

For the AOKR experiments [7, 8], with an initially Gaussian momentum distribution with root mean square > 1, the distribution of quasimomenta $f_0(\beta)$ is nearly uniform [10]. Since β is a constant of motion, the fidelity is

$$F(N) \equiv \left|\int d\beta f_0(\beta)\sum_{n=-\infty}^{+\infty}\psi_\beta^{(1)*}(N,n)\psi_\beta^{(2)}(N,n)\right|^2 .$$

To compute $F(N)$ we must calculate the overlap of two wave functions (3) for two values of k, and sum over all integer momenta n. For the case when the state of the atom is a plane wave with momentum $p_0 = n_0 + \beta$, we have $|\psi_\beta(\theta - N\xi - \arg(W_N))|^2 = 1/(2\pi)$, leading to the result for the fidelity for *one* β-state: $F_\beta(N) = J_0^2\left(|W_N|(k_1 - k_2)\right)$. For $\Delta k \equiv k_1 - k_2 = 0$, $F_\beta = F_\beta(N) = 1$. For resonant β we have $|W_N| = N$, and we can use

the asymptotic expansion formula [9.2.1] from [16] for the 0th order Bessel function J_0. The fidelity for large times at quantum resonance is then

$$F_{\beta_{\rm res}}(N) \simeq \frac{2}{\pi \Delta k N} \cos^2\left(\Delta k N - \frac{\pi}{4}\right) \ . \tag{8}$$

This shows that the fidelity falls off as a power law $\propto 1/N$ for the resonant β-states. Another particular case is given for $\beta = 0$: $F_{\beta=0}(N) = J_0^2\left(|\sin(N\pi/2)|\Delta k\right)$, which at $N = 1$ depends just on Δk. Then the fidelity can be made arbitrarily small when $\Delta k \simeq z_0$, where $z_0 \simeq 2.40$ is the first zero of J_0. In turn, the fidelity can be maximised by maximising J_0^2, what provides an option for coherent control when quasimomentum can be precisely controlled in experiments with Bose-Einstein condensates [9].

To compute $F(N)$ for $f_0(\beta) \equiv 1$, what approximates very well the experimental situation in [8, 10], we perform the average over the different quasimomenta, i.e.

$$\int_0^1 d\beta J_0\left(|W_N(\xi)|\Delta k\right) = \int_{-\pi}^{\pi} \frac{dx}{2\pi} J_0\left(\Delta k \sin(Nx)\csc(x)\right)$$

$$= \int_{-\pi}^{\pi} \frac{dx}{4\pi^2} \sum_{r=0}^{N-1} \frac{2\pi}{N} J_0\left(\Delta k \sin(x)\csc(xN^{-1} + 2\pi r N^{-1})\right) \ . \tag{9}$$

We used the definition of $W_N(\xi)$ as given after eq. (3), and the periodicity of the sine in the argument of the Bessel function. In the limit when $N \to \infty$ and $2\pi r/N \to \alpha$, the sum over r approximates the integral over α, and (9) converges to

$$\int_0^1 d\beta J_0\left(|W_N(\xi)|\Delta k\right) \to \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dx \int_0^{2\pi} d\alpha \ J_0\left(\Delta k \sin(x)\csc(\alpha)\right) \ .$$

With the help of [11.4.7] from [16], this leads us to the final result for the asymptotic value F^* of the fidelity [11]

$$F^*(\Delta k) \equiv \frac{1}{(2\pi)^2} \left(\int_0^{2\pi} d\alpha \ J_0^2\left(\frac{\Delta k \csc(\alpha)}{2}\right)\right)^2 \ . \tag{10}$$

The dependence on Δk of the formula (10) is shown in Fig. 2, together with numerical computations. $F^*(\Delta k)$ oscillates quasiperiodically rather than to drop monotonously with Δk. Note that the fidelity always saturates at exact resonance $\tau = 2\pi\ell$ ($\ell \in \mathbb{N}$), and the saturation time and level depend on Δk, as illustrated in the inset. We interprete our analytical result (10) by referring to the wave function in eq. (3): the perturbation enters in the expression for the fidelity in the form of phase factors $\exp\left(-\mathrm{i}\Delta k |W_N| \cos(\theta)\right)$. For non-resonant β, $|W_N| \to 0$ in the limit $N \to \infty$, and hence these phases eventually tend to unity, *i.e.*, the perturbation completely looses its effect on the relative evolution of the two wave packets under comparison. Note that a fidelity plateau, somewhat reminiscent of the saturation observed here, was recently predicted for some model systems [4]. However, this strictly semiclassical result was obtained in the perturbative limit, and the plateau observed in [4] only represents a transient effect, followed by the subsequent decay of fidelity. By contrast, our formula (10) is a result valid for all (sufficiently large) N.

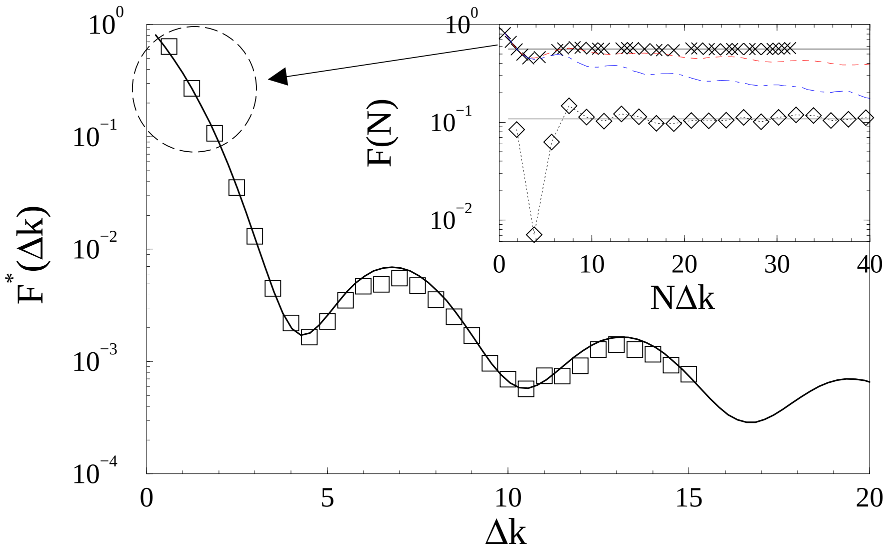

Figure 2. Asymptotical formula for the fidelity for $N \to \infty$ from eq. (10) (solid line), compared with numerical data (squares) obtained by evolving ensembles of 10^4 $\beta-$rotors with an initially uniform momentum distribution in $[0, 1)$, at $\tau = 2\pi$ and $N = 50$. Inset: numerically computed fidelity vs. $N\Delta k$ for $\tau = 2\pi$, $k_1 = 0.8\pi$, and fixed $\Delta k = 0.6283$ (crosses), 1.885 (diamonds). The position of the minima corresponds to the time $3.83\tau/\Delta k$. The fidelity saturates for times $N \gtrsim 20$ at Δk-dependent, constant values, indicated by the horizontal lines. Data for finite detunings $\epsilon = 0.025$ (dashed) and 0.1 (dot-dashed) is shown for $\Delta k = 0.6283$.

The asymptotic saturation value F^* depends on the difference Δk in the kick strength and, in addition, on the distribution of the initial conditions $f_0(\beta)$: the resonant β – which behave according to (8) – determine the initial decay and the consecutive rise of the fidelity $F(N)$ shown in the inset of Fig. 2. Their contribution quickly decreases, since the speed of the wings of the two wave packets to be compared is proportional to the two different k [5, 10]. After this initial stage, the bulk of non-resonant β takes over and dominates then the fidelity, leading to rapid saturation. These competing effects give rise to the damped oscillations around the asymptotic value of fidelity. At fixed Δk, F^* is the lower, the more weight is given to quasimomenta which are close to resonant values. The initial decay of the fidelity is due to nearly resonant β values: the minima in the inset in Fig. 2 are found by differentiating the time-dependent fidelity with respect to Δk. Then, the first zero of the 1st order Bessel function $J_1(z)$ ($z = z_1 \simeq 3.83$) is responsible for the observed minima. Figure 2 also shows that strict saturation is destroyed by arbitrarily small detunings $\epsilon \equiv 2\pi\ell - \tau$ from resonance (since then the quasiperiodic motion for the non-resonant β is destroyed). However, the temporal asymptotic decay of the fidelity depends continuously on ϵ, and is very slow for small ϵ, what leaves the here predicted saturation an *experimentally robust* observable (since the kicking period is the best controlled parameter in the experiment [6, 7, 8]).

5. Conclusions

In summary, we analytically derived the asymptotic behaviour of fidelity for kicked cold atoms at quantum resonance, far from the classical limit and for arbitrary perturbations in the kicking strength k. The predicted saturation sets in after a small number of kicks, at an experimentally observable level. Note, in particular, that $F^* \geq 50$ % for $\Delta k \sim O(k)$, what clearly stresses the robustness of quantum resonant motion. The type of dynamics occurring in-between the two interferometric pulses (see Fig. 1) is not restricted to the AOKR. Our derivation [see eqs. (4-7)] applies for arbitrary evolutions of cold atoms in a periodical potential, since it properly takes into account the different quasimomentum classes (corresponding to all the initial conditions) of the experimental ensemble.

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